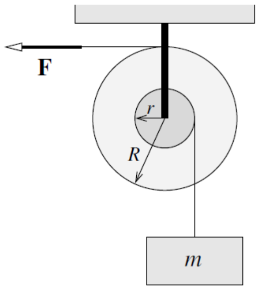
P1 [2009-2010 Fall - Final]: The device shown in the figure is used to lift a box of mass 30 kg. The outer radius of the device is 0.5 m and the radius of the hub is 0.2 m. When a 140-N force is applied to a rope wrapped around the outside of the device, the box has an upward acceleration of magnitude 0.8 m/s^2. Find the rotational inertia of the device.



SOLUTION: $m = 30\text{ kg}$, $R = 0.5\text{ m}$, $r = 0.2\text{ m}$, $F = 140\text{ N}$, $a = 0.8\text{ m/s}^2$

The tension T in the rope is found from a simple free-body diagram as

$$T - mg = ma \quad \Rightarrow T = m(a+g)$$

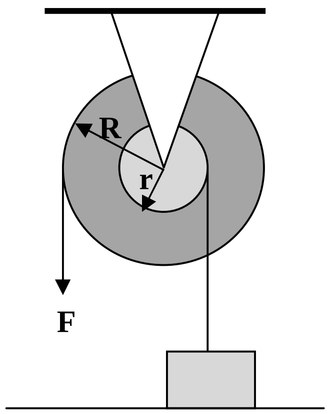
The angular acceleration is found from $\alpha = a/r$. Then, the application of Newton's 2nd law, $\tau_{\text{net}} = I\alpha$, gives

$$I = \frac{\tau_{\text{net}}}{\alpha} = \frac{FR - Tr}{\alpha} = \frac{FR - m(a+g)\cdot r}{a/r} = \frac{FRr - m(a+g)r^2}{a}$$

$$= \frac{(140)(0.5)(0.2) - (30)(0.8+9.8)(0.2)^2}{0.8} = 1.60\text{ kg}\cdot\text{m}^2$$ ☺

....

P2 [2010-2011 Fall - Final]: The pulley in the figure is used to lift a 40-kg box off the ground. The outer radius R of the pulley is 0.5 m and the inner radius r is 0.2 m. When a constant force of 200 N is applied to a rope wrapped around the pulley, the box has an upward linear acceleration of 0.3 m/s^2. (a) What is the rotational inertia of the pulley? (b) What is the angular velocity of the pulley 2 s later? (c) What is the angular momentum of the pulley at that time (2 s later)? (d) **(BONUS: +5 points)** What is the time dependent form of the kinetic energy of the pulley?



SOLUTION: $m = 40\text{ kg}$, $R = 0.5\text{ m}$, $r = 0.2\text{ m}$, $F = 200\text{ N}$, $a = 0.3\text{ m/s}^2$, $t = 2\text{s}$

(a) We first find the tension in the rope. It follows from a simple FBD that

$$T - mg = ma \quad \Rightarrow \quad T = m(a+g)$$

The net torque due to the tension and the applied force is

$$\tau_{\text{net}} = FR - Tr = FR - m(a+g)r$$

It is also required that

$\tau_{\text{net}} = I\alpha$,

where

$\alpha = a/r = 0.3/0.2 = 1.5 \text{ rad/s}^2$.

Thus, we find the rotational inertia of the pulley as

$$I = \frac{\tau_{\text{net}}}{\alpha} = \frac{FR - m(a+g)r}{a/r} = \frac{(200)(0.5)-(40)(0.3+9.8)(0.2)}{1.5} = 12.8 \text{ kg}\cdot\text{m}^2$$

(b) $\omega = \underbrace{\omega_0}_{=0} + \alpha t = \alpha t = (1.5)(2) = 3.00 \text{ rad/s}$ ☺

(c) $L = I\omega = (12.8)(3) = 38.4 \text{ kg}\cdot\text{m}^2/\text{s}^2$ (counterclockwise) ☺

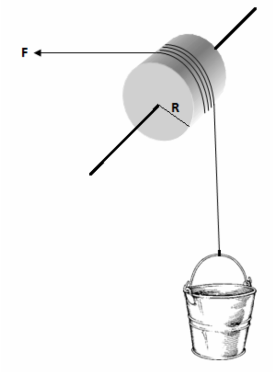
(d) $\omega(t) = \underbrace{\omega_0}_{=0} + \alpha t = \alpha t$

$$K(t) = \tfrac{1}{2} I\,\omega^2(t) = \tfrac{1}{2} I \cdot (\alpha t)^2 = \tfrac{1}{2} I\alpha^2 t^2 = \left(\tfrac{1}{2}\right)(12.8)(1.5)^2 t^2 \quad \Rightarrow \quad K(t) = 14.4 t^2$$

where K is in joules and t is in seconds. ☺

....

P3 [2010-2011 Fall - Final]: A bucket of mass $m = 1 \text{ kg}$ is pulled by a force $\vec{F}$ of magnitude 20 N via a cord of negligible mass. The cord is wrapped around a cylinder of mass $M = 8.2 \text{ kg}$ and radius $R = 10 \text{ cm}$. Starting from rest, the bucket is raised 9 m in 3 s . (a) Find the bucket's linear acceleration and the cylinder's angular acceleration. (b) What is the net torque acting on the cylinder? (c) Calculate the rotational inertia of the cylinder. (d) How much work is done on the cylinder to raise the bucket to the 9-m height?



SOLUTION: $m = 1 \text{ kg}$, $F = 20 \text{ N}$, $M = 8.2 \text{ kg}$, $R = 10 \text{ cm} = 0.1 \text{ m}$, $h = s = 9 \text{ m}$, $t = 3 \text{ s}$

(a) $h = \tfrac{1}{2} a t^2 \quad \Rightarrow \quad a = \dfrac{2h}{t^2} = \dfrac{(2)(9)}{3^2} \quad \Rightarrow \quad a = 2.00 \text{ m/s}^2$ ☺

$a = R\alpha \quad \Rightarrow \quad \alpha = \dfrac{a}{R} = \dfrac{2}{0.1} \quad \Rightarrow \quad \alpha = 20.0 \text{ rad/s}^2$ ☺

(b) Making use of a simple FBD, the expression for the tension in the rope is

$T - mg = ma \quad \Rightarrow \quad T = m(a+g)$

The net torque due to the tension and the applied force is

$$\tau_{\text{net}} = FR - TR = FR - m(a+g)R = R[F - m(a+g)]$$

$$\Rightarrow \quad \tau_{\text{net}} = (0.1)[20 - (1)(2+9.8)] = 0.820 \text{ N}\cdot\text{m} \qquad ☺$$

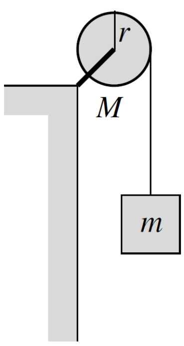
(c) $\tau_{\text{net}} = I\alpha \quad \Rightarrow \quad I = \dfrac{\tau_{\text{net}}}{\alpha} = \dfrac{0.82}{20} = 0.0410 \text{ kg}\cdot\text{m}^2$ ☺

(d) $s = R\Delta\theta \quad \Rightarrow \quad \Delta\theta = \dfrac{s}{R}$

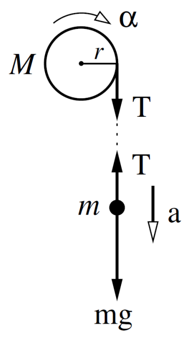
$$W = \tau_{\text{net}}\,\Delta\theta = \tau_{\text{net}}\,\frac{s}{R} = (0.82)\frac{9}{0.1} \quad \Rightarrow \quad W = 73.8 \text{ J} \qquad ☺$$

....

P4 [2010-2011 Summer - MT3]: A disk of radius $r = 11$ cm and mass $M = 2.5$ kg is initially at rest. A rope of negligible mass is wrapped around the disk at one end, and at the other end is a block of mass $m = 1.7$ kg. Now the disk is set to rotate freely. (a) Draw the free body diagrams of the system. Find (b) the acceleration of block m, (c) the tensional force on the rope, (d) the torque exerted by the rope on the disk, (e) the angular acceleration of the disk, and (f) the angular velocity of the cylinder at $t = 5$ s. (The rotational inertia of the disk is $I = \frac{1}{2}Mr^2$.)



SOLUTION: $r = 0.11$ m, $M = 2.5$ kg, $m = 1.7$ kg, $\omega_0 = 0$, $t = 5$ s



(b) $\left.\begin{array}{l} \tau = Tr = I\alpha \\ mg - T = ma \quad \Rightarrow \quad T = mg - ma \\ a = \alpha r \end{array}\right\} \quad (mg - ma)r = \frac{1}{2}Mr^2 \cdot \dfrac{a}{r} \quad \Rightarrow \quad mg - ma = \dfrac{Ma}{2}$

$$\Rightarrow \quad a = \frac{2mg}{M + 2m} = \frac{(2)(1.7)(9.8)}{2.5 + (2)(1.7)} = 5.6475 \text{ m/s}^2 \approx 5.65 \text{ m/s}^2 \qquad ☺$$

(c) $T = mg - ma = m(g - a) = (1.7)(9.8 - 5.6475) = 7.0593 \text{ N} \approx 7.06 \text{ N}$

(d) $\tau = Tr = (7.0593)(0.11) = 0.77652 \text{ N}\cdot\text{m} \approx 0.777 \text{ N}\cdot\text{m}$ ☺

(e) $\tau = I\alpha = \quad \Rightarrow \quad \alpha = \dfrac{\tau}{I} = \dfrac{\tau}{Mr^2/2} = \dfrac{2\tau}{Mr^2} = \dfrac{(2)(0.77652)}{(2.5)(0.11)^2} = 51.340 \text{ rad/s}^2 \approx 51.3 \text{ rad/s}^2$ ☺

(f) $\omega = \omega_0 + \alpha t = 0 + (51.340)(5) \approx 257 \text{ rad/s}$ ☺

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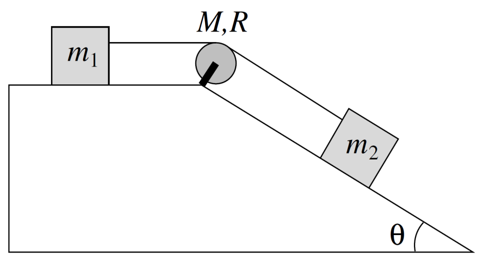
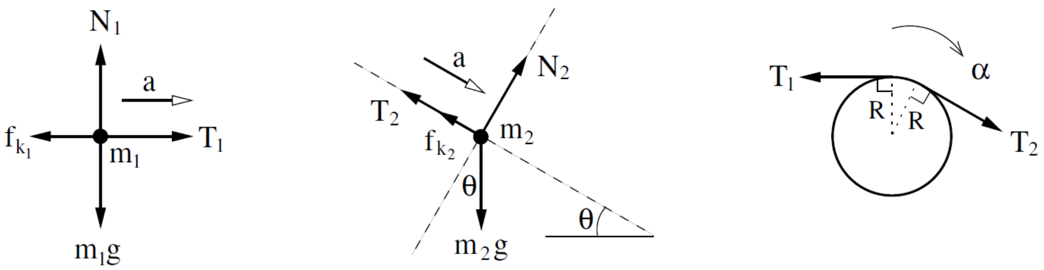
P5 [2010-2011 Summer - Final]: A block of mass $m_1 = 2$ kg and a block of mass $m_2 = 6$ kg are connected by a massless string over a pulley in the shape of a solid disk having radius $R = 0.25$ m and mass $M = 10$ kg. These blocks are allowed to move on a fixed block-wedge of angle $\theta = 30^{\circ}$. The coefficient of kinetic friction is $\mu_k = 0.36$ for both blocks. Draw free-body diagrams of both blocks and of the pulley. (a) Draw the free-body diagrams for the blocks and the disk. Determine (b) the acceleration of the two blocks and (c) the tensions in the string on both sides of the pulley. (The rotational inertia of the disk is $I = \frac{1}{2}MR^2$.)



SOLUTION: $m_1 = 2$ kg, $m_2 = 6$ kg, $R = 0.25$ m, $M = 10$ kg, $\theta = 30^{\circ}$, $\mu_k = 0.36$



(b) Equations of motion for m_1:

$N_1 = m_1 g$ (1)

$T_1 - f_{k_1} = m_1 a$ (2)

with

$f_{k_1} = \mu_k N_1$ (3)

Equations of motion for m_2:

$N_2 = m_2 g \cos\theta$ (4)

$m_2 g \sin\theta - T_2 - f_{k_2} = m_2 a$ (5)

with

$f_{k_2} = \mu_k N_2$ (6)

Equation of motion for the disk:

$\tau_{net} = T_2 R - T_1 R = I\alpha$ (7)

with

$a = \alpha R$ (8)

and

$I = \frac{1}{2} M R^2$ (9)

With these, Eqs.(2), (5), and (7) respectively become

$T_1 - \mu_k m_1 g = m_1 a$ (9)

$m_2 g \sin\theta - T_2 - \mu_k m_2 g \cos\theta = m_2 a$ (10)

$T_2 - T_1 = \frac{1}{2} M a$ (11)

Adding these side-by-side, we obtain the acceleration as

$$a = \frac{\left[m_2 \sin\theta - \mu_k \left(m_1 + m_2 \cos\theta\right)\right] g}{m_1 + m_2 + \frac{1}{2} M}$$

$$\Rightarrow \quad a = \frac{\left[6 \sin 30^{\circ} - (0.36)\left(2 + 6 \cos 30^{\circ}\right)\right](9.8)}{2 + 6 + \left(\frac{1}{2}\right)(10)} = 0.30861 \text{ m/s}^2 \approx 0.309 \text{ m/s}^2$$ ☺

(c) From Eq.(9)

$T_1 = m_1 \left(\mu_k g + a\right) = (2)\left[(0.36)(9.8) + 0.30861\right] = 7.6632 \text{ N} \approx 7.66 \text{ N}$ ☺

From Eq.(11)

$T_2 = \frac{1}{2} M a + T_1 = \left(\frac{1}{2}\right)(10)(0.30861) + 7.6632 \approx 9.21 \text{ N}$ ☺

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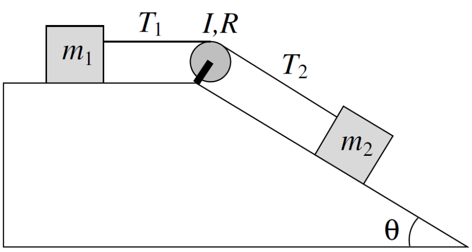
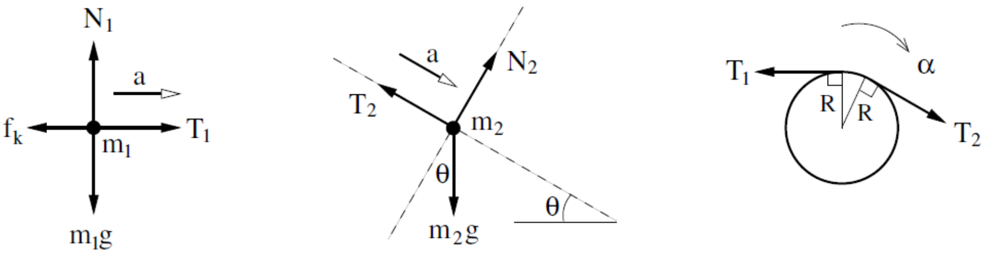
P6 [2011-2012 Spring - Final]: Two blocks of masses $m_1 = 1.5$ kg and $m_2 = 4$ kg are connected by a string that passes over a frictionless pulley, whose radius is $R = 7$ cm and whose rotational inertia is $I = 0.38 \text{ kg} \cdot \text{m}^2$. Horizontal surface has coefficient of kinetic friction $\mu_k = 0.4$ but the inclined plane, with $\theta = 35^{\circ}$, is frictionless. The system is released from rest. (a) Draw the free-body-diagram of the blocks and the pulley. (b) Find the linear (translational) acceleration of the blocks. (c) Find the angular acceleration of the pulley. (d) Find the tensions T_1 and T_2.



SOLUTION: $m_1 = 1.5$ kg, $m_2 = 4$ kg, $R = 7 \text{ cm} = 0.07 \text{ m}$, $I = 0.38 \text{ kg} \cdot \text{m}^2$, $\mu_k = 0.4$, $\theta = 35^{\circ}$

(a) The free-body diagrams are shown below.



(b) Equations of motion for m_1:

$N_1 = m_1 g$ (1)

$T_1 - f_k = m_1 a$ (2)

with

$f_k = \mu_k N_1$ (3)

Equations of motion for m_2:

$N_2 = m_2 g \cos\theta$ (4)

$m_2 g \sin\theta - T_2 = m_2 a$ (5)

Equation of motion for the disk:

$\tau_{net} = T_2 R - T_1 R = I\alpha$ (6)

with

$a = \alpha R$ (7)

With these , Eqs.(2) and (6) respectively become

$T_1 - \mu_k m_1 g = m_1 a$ (8)

and

$(T_2 - T_1)R = \frac{I}{R} a \;\Rightarrow\; T_2 - T_1 = \frac{I}{R^2} a$ (9)

Add Eqs.(5), (8), and (9) side-by-side to obtain the acceleration as

$$a = \frac{(m_2 \sin\theta - \mu_k m_1)g}{m_1 + m_2 + I/R^2} = \frac{[4\sin 35^\circ - (0.4)(1.5)](9.8)}{1.5 + 4 + 0.38/(0.07)^2} = 0.19993\ \text{m/s}^2 \approx 0.200\ \text{m/s}^2$$ ☺

(c) $\alpha = \frac{a}{R} = \frac{0.19993}{0.07} \approx 2.86\ \text{rad/s}$

(d) From Eq.(8),

$$T_1 = m_1(\mu_k g + a) = (1.5)[(0.4)(9.8) + 0.19993] = 6.1799\ \text{N} \approx 6.18\ \text{N} \qquad ☺$$

and from Eq.(9),

$$T_2 = \frac{I}{R^2}a + T_1 = \frac{0.38}{(0.07)^2}(0.19993) + 6.1799 \approx 21.7\ \text{N} \qquad ☺$$

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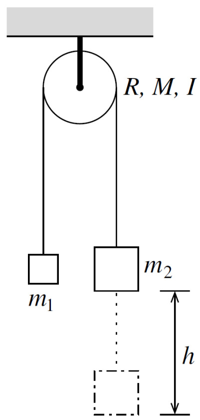
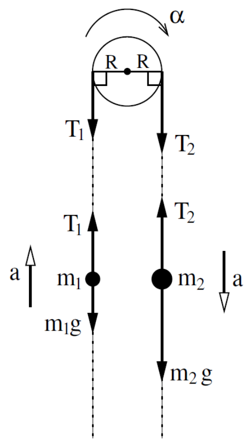
P7 [2013-2014 Spring - Final]: The Atwood machine in the figure is initially at rest. The masses of the hanging blocks are $m_1 = 2.9$ kg and $m_2 = 4.7$ kg. As to the pulley, its mass is $M = 2.1$ kg, its radius is $R = 11$ cm, and its rotational inertia is $I = \frac{1}{2}MR^2$. When the system is set free, the block m_2 descends a distance $h = 40$ cm. Using Newton's 2nd law for rotating bodies, calculate at this moment: (a) the linear acceleration of block m_2, (b) the angular acceleration of the pulley, (c) the tensions in the rope, and (d) the speed of block m_2. (e) [**BONUS, 10 points**] Find the speed of block m_2 *by making use of energy considerations.*



SOLUTION: $m_1 = 2.9$ kg, $m_2 = 4.7$ kg, $M = 2.1$ kg, $R = 11\ \text{cm} = 0.11$ m, $h = 40\ \text{cm} = 0.4$ m, $I = \frac{1}{2}MR^2$

(a) Check the following free-body diagrams of the two blocks and the pulley. Equations of motion are



$$m_2 g - T_2 = m_2 a \qquad (1)$$

$$T_1 - m_1 g = m_1 a \qquad (2)$$

$$T_2 R - T_1 R = I\alpha \qquad (3)$$

with

$$\alpha = \frac{a}{R} \qquad (4)$$

and

$$I = \tfrac{1}{2}MR^2 \qquad (5)$$

With (4) and (5), (3) becomes

$$(T_2 - T_1)R = \tfrac{1}{2}MR^2 \cdot \frac{a}{R} \ \Rightarrow\ T_2 - T_1 = \tfrac{1}{2}Ma \qquad (6)$$

Add (1) and (2) side-by-side and use (6):

$$\underbrace{T_1 - T_2}_{-\frac{1}{2}Ma} + (m_2 - m_1)g = (m_1 + m_2)a \;\Rightarrow\; (m_2 - m_1)g = (m_1 + m_2)a + \tfrac{1}{2}Ma$$

whence

$$a = \frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M}g = \frac{4.7 - 2.9}{2.9 + 4.7 + \left(\frac{1}{2}\right)(2.1)}(9.8) = 2.0393 \text{ m/s}^2 \approx 2.04 \text{ m/s}^2 \qquad ☺$$

(b) $\alpha = \dfrac{a}{R} = \dfrac{2.0393}{0.11} \approx 18.5 \text{ rad/s}^2$ ☺

(c) Using (2) and (1):

$$T_1 = m_1(g + a) = (2.9)(9.8 + 2.0393) \approx 34.3 \text{ N} \qquad ☺$$

$$T_2 = m_2(g - a) = (4.7)(9.8 - 2.0393) \approx 36.4 \text{ N} \qquad ☺$$

(d) $v^2 = \underbrace{v_0^2}_{0} + 2ah \;\Rightarrow\; v = \sqrt{2ah} = \sqrt{(2)(2.0393)(0.4)} \approx 1.28 \text{ m/s}$ ☺

(e) Take the initial levels of the blocks as being the zero-potential level. A rudimentary energy conservation calculation yields

$$0 = m_1gh - m_2gh + \tfrac{1}{2}m_1v^2 + \tfrac{1}{2}m_2v^2 + \tfrac{1}{2}I\omega^2$$

Using $I = \frac{1}{2}MR^2$ and $v = R\omega$, we get

$$0 = (m_1 - m_2)gh + \tfrac{1}{2}(m_1 + m_2)v^2 + \tfrac{1}{2}\cdot\tfrac{1}{2}MR^2\cdot\omega^2 = (m_1 - m_2)gh + \tfrac{1}{2}(m_1 + m_2)v^2 + \tfrac{1}{4}Mv^2$$

or

$$0 = 4(m_1 - m_2)gh + 2(m_1 + m_2)v^2 + Mv^2$$

whence

$v = \sqrt{\dfrac{4(m_2 - m_1)gh}{2(m_1 + m_2) + M}} = \sqrt{\dfrac{(4)(4.7 - 2.9)(9.8)(0.4)}{(2)(2.9 + 4.7) + 2.1}} \approx 1.28 \text{ m/s}$, as before. ☺

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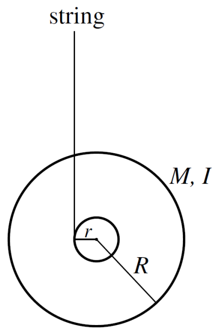
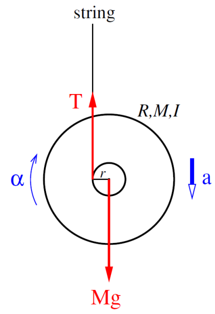
P8 [2014-2015 Fall - Final]: Consider a nearly cylindrical yo-yo like shown in the figure. Its mass is M, its outer radius is R, and its axle's radius is $r = R/5$. As the yo-yo descends, what are (a) its linear acceleration and (b) the tension in the string. (c) If the yo-yo starts from rest and falls down freely, what is its speed when it descends a vertical distance h? (The rotational inertia of the yo-yo about its center of mass is $I = \frac{1}{2}MR^2$.)



SOLUTION: $r = R/5$, $I = \frac{1}{2}MR^2$

(a) Check the free-body diagram below. The equation of motion for the vertical translational motion is



$F_{\text{net}} = Mg - T = Ma$ (1)

and that for the rotational motion is

$\tau_{\text{net}} = Tr = I\alpha$ (2)

where we have noted that the weight Mg of the yo-yo does not produce any torque since its direction passes through the rotational axis of the yo-yo. We advance (2) as

$$Tr = \tfrac{1}{2}MR^2 \cdot \frac{a}{r} \Rightarrow T = \tfrac{1}{2}MR^2 \frac{a}{r^2}$$

Using $R = 5r$, we have

$$T = \tfrac{1}{2}M(5r)^2 \frac{a}{r^2} = \tfrac{25}{2}Ma \quad (3)$$

Inserting (3) into (1), we obtain the desired acceleration as

$$Mg - T = Ma \Rightarrow Mg - \tfrac{25}{2}Ma = Ma \Rightarrow g - \tfrac{25}{2}a = a \Rightarrow g = \left(1 + \tfrac{25}{2}\right)a = \tfrac{27}{2}a$$

$\therefore\ a = \frac{2}{27}g$ (4) ☺

(b) Using (4) in (3), we get

$$T = \tfrac{25}{2}M \cdot \tfrac{2}{27}g \Rightarrow T = \tfrac{25}{27}Mg \quad ☺$$

(c) $v^2 = \underbrace{v_0^2}_{=0} + 2ah = 2ah = 2 \cdot \frac{2}{27}g \cdot h = \frac{4}{27}gh \Rightarrow v = \sqrt{\frac{4}{27}gh}$ ☺

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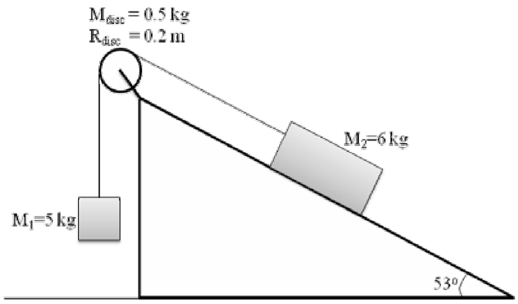
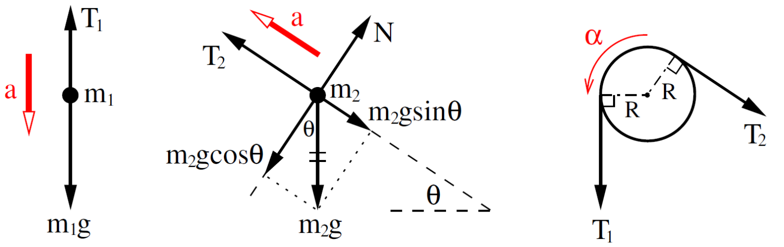
P9 [2014-2015 Fall - Final]: The system given in the figure is initially at rest and there is no friction in any part. The system is now released. (a) Find the acceleration of the blocks. (b) Calculate the tensions in the cord. (The rotational inertia of the disc about its center of mass is $I = \frac{1}{2}MR^2$.)



SOLUTION: $m_1 = 5\text{ kg}$, $m_2 = 6\text{ kg}$, $M = 0.5\text{ kg}$, $R = 0.2\text{ m}$, $\theta = 53^\circ$, $I = \frac{1}{2}MR^2$

(a) The free-body diagrams are shown below:



The corresponding equations of motion are

$$m_1 g - T_1 = m_1 a \qquad (1)$$

$$T_2 - m_2 g \sin\theta = m_2 a \qquad (2)$$

$$T_1 R - T_2 R = I\alpha \;\Rightarrow\; (T_1 - T_2)R = \tfrac{1}{2}MR^2 \cdot \frac{a}{R} \;\Rightarrow\; T_1 - T_2 = \tfrac{1}{2}Ma \qquad (3)$$

Add these three equations side-by-side:

$$m_1 g - m_2 g \sin\theta = m_1 a + m_2 a + \tfrac{1}{2}Ma \;\Rightarrow\; (m_1 - m_2\sin\theta)g = \left(m_1 + m_2 + \tfrac{1}{2}M\right)a$$

$$\therefore\; a = \frac{m \;- m_2 \sin\theta}{m_1 + m_2 + \frac{1}{2}M}g = \frac{5 - 6\sin 53^\circ}{5 + 6 + \left(\frac{1}{2}\right)(0.5)}(9.8) = 0.18135\text{ m/s}^2 \approx 0.181\text{ m/s}^2$$ ☺

(b) Tensions follow from (1) and (2):

$$T_1 = m_1(g - a) = (5)(9.8 - 0.18135) = 48.093\text{ N}$$ ☺

$$T_2 = m_2(a + g\sin\theta) = (6)\left(0.18135 + 9.8\sin 53^\circ\right) = 48.048\text{ N}$$ ☺

Note that the magnitude two tensions are so close to each other that we have expressed them in 5-digit form.

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