# CHAPTER 3 | VECTORS

Hüseyin Oymak
Ümit Alkuş

Atılım University
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# Outline





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# Scalars

- If a single number is sufficient to describe a (physical) quantity, then that quantity is said to be a **scalar quantity**.
- Examples are volume, mass, energy, heat, temperature, electric current, etc.

# Vectors

- In physics and in all engineering sciences there are a large number of quantities such as position, displacement, velocity, and acceleration that possess both *magnitude* and *direction*.
- We geometrically represent them by arrows and denote by $\overrightarrow{P_1P_2}$ with the understanding that tail is at the point $P_1$ and head is at the point $P_2$.

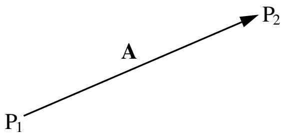



Figure 1: A vector.

# Vectors

- In handwriting we put an arrow over a letter to denote a vector; in print, we use a single letter written in boldface type. For example, the vector in Figure 1 is denoted by either of the notations

$$\vec{A} = \overrightarrow{P_1P_2} \qquad \text{or} \qquad \mathbf{A} = \overrightarrow{P_1P_2}$$

- We shall almost always use the latter notation, **A**, to represent a vector.
- The *magnitude* of a vector **A** is the length of the arrow (or the size of the quantity which is represented by the vector **A**) and is denoted by one of the following notations:

$$A \equiv |\mathbf{A}| \equiv |\overrightarrow{P_1P_2}| \qquad \text{(in print)}$$
$$A \equiv |\vec{A}| \equiv |\overrightarrow{P_1P_2}| \qquad \text{(in handwriting)}$$

# Displacement Vector

- The vector $\mathbf{A} = \overrightarrow{P_1P_2}$ might represent the **displacement vector** of an object that moves from point $P_1$ to point $P_2$, using *any* path between these two points, as shown in Figure 2.
- Therefore, the displacement vector does *not* give any information about the actual path of the object; it tells us only about the *change of position* of the object moving between these two definite points.

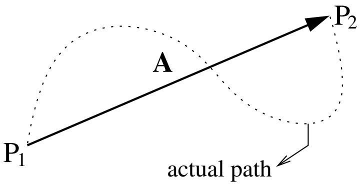



Figure 2: A displacement vector.

# Null Vector

- We define the **null vector**, or **zero vector**, denoted by **0**, as having zero length and no specific direction.

# Equal Vectors

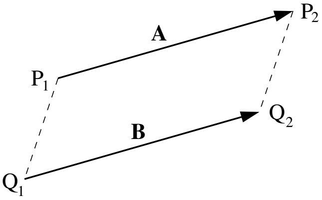



Figure 3: Two equal vectors.

- When two vectors **A** and **B** have the same length and the same direction, as shown in Figure 3, they are said to **equal** each other.
- The exact positions are not important (exception is the position vector.
- *transportable: you can put a vector wherever you like in a coordinate system, and then you can slide it completely arbitrarily, parallel to itself, to another place.*

# Vector Summation

- Vectors can be summed together to obtain another vector, which is called their (**vector) sum** or their **resultant**.
- Consider two vectors **A** and **B**, this summation is performed geometrically by placing the tail of one vector at the head of the other.

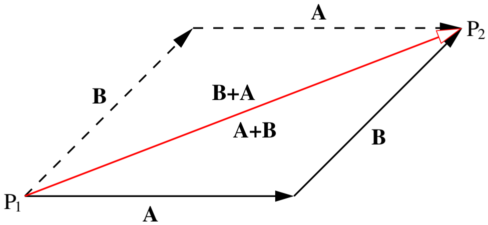



Figure 4: Summation of two vectors. This figure shows also that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

- You can add only vectors of the same kind. You *cannot* add a force vector, for example, to a velocity vector.

# Vector Summation

- If **A** and **B** are two displacement vectors for an object moving from point $P_1$ to point $P_2$ following two different paths, their sum $\mathbf{A}+\mathbf{B}$ gives the *overall* or *net* displacement of the object. Figure 4 exemplifies also such a case.
- We said the summation of two vectors **A** and **B** gives another vector, say **C**. We write mathematically the relation among them as a **vector equation**

$$\mathbf{C}=\mathbf{A}+\mathbf{B}$$

- Although there appears deceptively a single equation here, it includes actually three distinct equations, one for each dimension $x$, $y$, and $z$, as you will notice clearly later.

# Vector Summation

- It is an easy task to show that the summation operation is **commutative**; that is, the order of addition is of no importance (Figure 4):

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

- We also say that the summation operation is **associative**; that is, if there are three or more vectors to sum, we are free to pick and put any of them into a group and add them in any order we like:

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{C}) + \mathbf{B}$$

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{C}) + \mathbf{B}$$

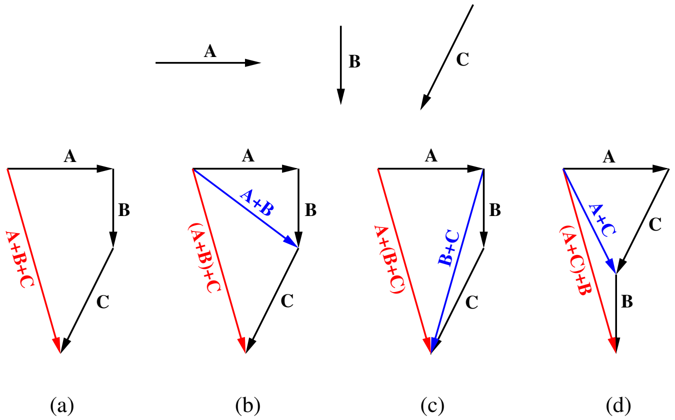



Figure 5: Summation of three vectors.

# Scalar Multiplication

- By definition, if a vector **A** is multiplied by a *scalar c*, the result is another vector, *c***A**.
- The magnitude of this new vector is $|c|A$.
- If $c > 0$, *c***A** is *parallel to* **A**; if $c < 0$, *c***A** is *antiparallel to* **A**.

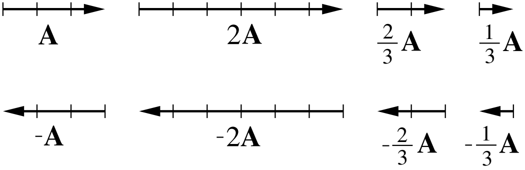



Figure 6: Multiplying a vector with a scalar.

- Scalar multiplication is *contraction* or *elongation* of a vector.

# Scalar Multiplication

- Multiplying a vector by $c = 0$, the result is the null vector.
- Multiplication by unity, $c = 1$, leaves the vector unchanged.
- Multiplication by $c = -1$ leads to the (**additive**) **inverse** of vector **A** denoted by $-\mathbf{A}$.
- It is obvious that the scalar multiplication is **distributive**:

$$c(\mathbf{A} + \mathbf{B}) = (\mathbf{A} + \mathbf{B})c = c\mathbf{A} + c\mathbf{B}$$

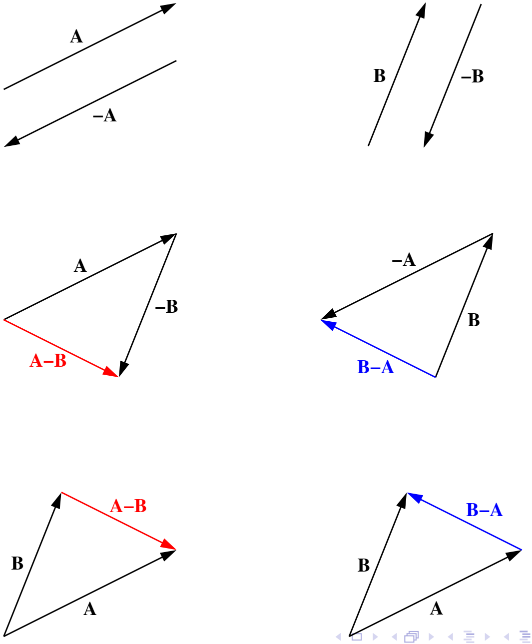
# Subtraction of Vectors

- We can **subtract** a vector **B** from another vector **A** as

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

- $\mathbf{A} - \mathbf{B}$ is the vector summation of **A** and the inverse vector $-\mathbf{B}$.

# Subtraction of Vectors



# Resolving a Vector Lying on an *xy*-Plane

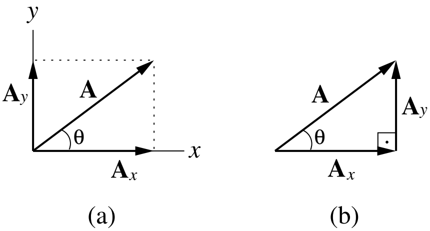
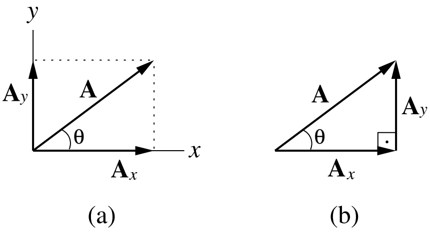



Figure 7: Components of a vector in an *xy*-plane.

- **Resolving** (or **Decomposing**) a vector into its **components**, you find its projections on the axes of a coordinate system.
- Consider a vector **A** lying on an *xy*-plane. Its projection on the *x*-axis is referred to as its *x-component*, $\mathbf{A}_x$, and that on the *y*-axis as its *y-component*, $\mathbf{A}_y$.

# Resolving a Vector Lying on an *xy*-Plane




- Since vectors are transportable, vector **A** is

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$$

- The magnitudes of the components of **A** are

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta$$

# Resolving a Vector Lying on an *xy*-Plane

- The relation between the three magnitudes in Figure 7 (b) is found by the Pythagorean theorem as

$$A = \sqrt{A_x^2 + A_y^2}$$

- Angle $\theta$ is determined from

$$\tan\theta = \frac{A_y}{A_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

- The **component notation:** $(A_x, A_y)$,
- The **magnitude-angle notation:** $(A, \theta)$

# Resolving a Vector Lying on an *xy*-Plane

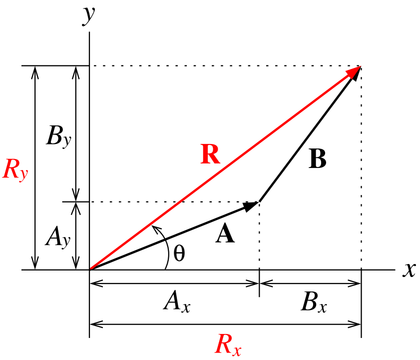
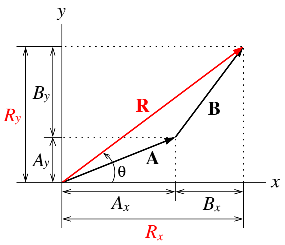



Figure 8: A vector and its components on an *xy*-plane.

- Pictorial summation of vectors.
- The summation: $\mathbf{R} = \mathbf{A} + \mathbf{B}$

# Resolving a Vector Lying on an *xy*-Plane




- The *x*- and *y*-components of the resultant vector **R**:

$$R_x = A_x + B_x \quad \text{and} \quad R_y = A_y + B_y$$

- Magnitude of **R**:

$$R = \sqrt{R_x^2 + R_y^2}$$

- The angle that **R** makes with the positive *x*-axis:

$$\tan\theta = \frac{R_y}{R_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

# Unit Vectors

- The **unit vector A** of the *nonzero* vector **A** is defined to have a length exactly 1 and to be in the same direction as **A**. The formula is

$$\mathbf{A} = \frac{\mathbf{A}}{A}$$

- The most important equation about vectors:

$$\mathbf{A} = A\mathbf{A}$$

- The magnitude and the direction of a vector are embodied in this equation, i.e., a vector is the product of its magnitude and its direction.



Hüseyin Oymak
Ümit Alkuş

# Unit Vectors

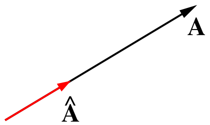



Figure 9: A vector with its unit vector pointing its direction.

- A unit vector does *not* have a dimension nor a unit.
- A unit vector only specifies a direction.

# Unit Vectors

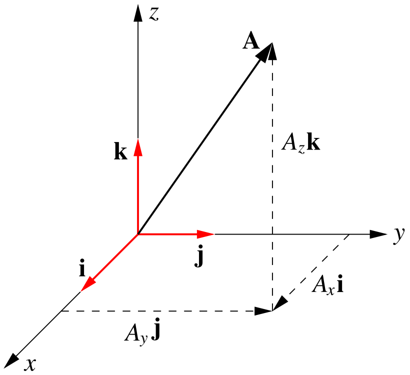



Figure 10: A vector and its components in terms of unit vectors.

- In the usual **right-handed rectangular** *xyz*-**coordinate system**, we shall use three special **cartesian unit vectors i**, **j**, and **k** that point the positive directions of the *x*-, *y*, and *z*-axes, respectively.

# Unit Vectors

- In handwriting these unit vectors appear as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.
- More specifically, the vector **i** is from origin to the point (1,0,0), **j** is from origin to the point (0,1,0), and **k** is from origin to the point (0,0,1).
- Any vector **A** is a linear combination of these **basis vectors**:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

# Unit Vectors

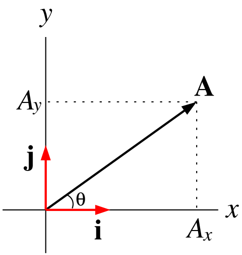



Figure 11: A vector its components on an *xy*-plane.

- Consider a 2D vector lying on the *xy*-plane. the relation $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$ reduces to

$$\begin{aligned}\mathbf{A} &= A_x\,\mathbf{i} + A_y\,\mathbf{j} \\ &= A\cos\theta\,\mathbf{i} + A\sin\theta\,\mathbf{j} \\ &= A(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\end{aligned}$$

# Unit Vectors

- Recalling $A_x = A\cos\theta$ and $A_y = A\sin\theta$ and using $\mathbf{A} = A\mathbf{A}$, the following relations are obtained

$$\mathbf{A} = A\mathbf{A} \quad \text{where} \quad \mathbf{A} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

- Recall Figure 8 and use unit-vector notation for the pictorial vector sum: $\mathbf{R} = \mathbf{A} + \mathbf{B}$:

$$R_x = A_x + B_x \quad \text{and} \quad R_y = A_y + B_y$$

$$\Rightarrow \mathbf{R} = R_x\,\mathbf{i} + R_y\,\mathbf{j} = (A_x + B_x)\,\mathbf{i} + (A_y + B_y)\,\mathbf{j}$$

# Unit Vectors

- The magnitude of **R** and the angle that **R** makes with the positive $x$-axis

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right)$$

# Vectors in Space: 3D Generalization

- Generalization of basic vector properties for 3D space vectors with the help of cartesian unit vectors **i**, **j**, and **k**.
- A 3D vector **A** as a linear combination of **i**, **j**, and **k**:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

- The magnitude (or length) of **A**:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

- The 3D **null** (**zero**) **vector**, **0**

$$\mathbf{0} = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$$

# Vectors in Space: 3D Generalization

- The **summation** and **subtraction** of two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ is given as

$$\begin{aligned} \mathbf{R} &= \mathbf{A} \pm \mathbf{B} \\ &= (A_x \pm B_x)\,\mathbf{i} + (A_y \pm B_y)\,\mathbf{j} + (A_z \pm B_z)\,\mathbf{k} \\ &= R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_y\,\mathbf{k} \end{aligned}$$

- The magnitude of the resultant vector **R**:

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2 + R_z^2} \\ &= \sqrt{(A_x \pm B_x)^2 + (A_y \pm B_y)^2 + (A_z \pm B_z)^2} \end{aligned}$$

# Vectors in Space: 3D Generalization

- **Scalar multiplication** of a 3D vector **A** by a **scalar** *c*, a real number:

$$\begin{aligned} c\mathbf{A} &= cA_x\,\mathbf{i} + cA_y\,\mathbf{j} + cA_z\,\mathbf{k} \\ |c\mathbf{A}| &= |c|A \end{aligned}$$

- The scalar multiplication is **distributive**:

$$c(\mathbf{A}+\mathbf{B}) = (\mathbf{A}+\mathbf{B})c = c\mathbf{A} + c\mathbf{B}$$

# Position Vector

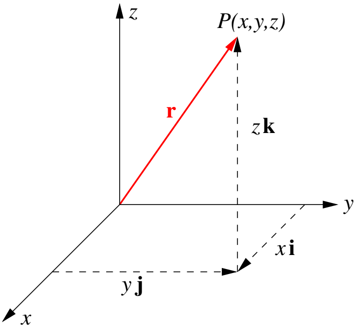



Figure 12: The position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

- Consider the **position vector**

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- Tail at the origin and head at the point $P(x, y, z)$.
- The position vector **r** is in general used to denote the position of a moving object at any time.

# Displacement Vector

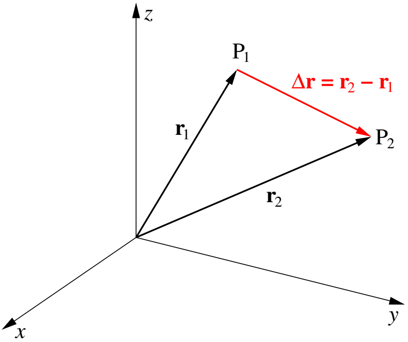



Figure 13: The displacement vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

- An object moving from $P_1$ to $P_2$.
- The net change in the position of the object is the so-called **displacement vector**, denoted by $\Delta\mathbf{r}$

# Displacement Vector

- From Figure 13, $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.
- Use the notion of vector subtraction:

$$\begin{aligned} \Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k} \end{aligned}$$

- Much more meaningful expression for $\Delta\mathbf{r}$:

$$\mathbf{r}_1 + \Delta\mathbf{r} = \mathbf{r}_2$$

- An object initially at position $\mathbf{r}_1$ undergoes a displacement $\Delta\mathbf{r}$, it arrives at position $\mathbf{r}_2$.

# The Scalar Product of Two Vectors

- Given two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, we define their **scalar product** $\mathbf{A} \cdot \mathbf{B}$ as

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

- The result is just a *scalar*, a real number.
- Due to the symbol "·" in its notation, the scalar product is also referred to as the **dot product**. The other common name is the **inner product**.

# The Scalar Product of Two Vectors

- Compare $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ to the equation $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$ putting $\mathbf{B} = \mathbf{A}$.
- The magnitude of vector **A** is then the square root of the scalar product $\mathbf{A} \cdot \mathbf{A}$:

$$A = \sqrt{\mathbf{A} \cdot \mathbf{A}} \qquad \text{or} \qquad A^2 = \mathbf{A} \cdot \mathbf{A}$$

# The Scalar Product of Two Vectors

- The scalar product of the standard basis vectors

$$\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$$

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{i}=\mathbf{i}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{j}=0$$

- The scalar product is commutative and distributive:

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A}\\
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= (\mathbf{B}+\mathbf{C})\cdot\mathbf{A}=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\\
c\,(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\mathbf{B})\,c=(c\,\mathbf{A})\cdot\mathbf{B}=\mathbf{A}\cdot(c\,\mathbf{B})
\end{aligned}$$

# Cosine Law

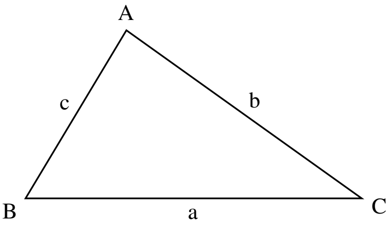



Figure 14: The triangle for the cosine law.

- For the triangle in Figure 14 the following formulas can be shown to hold:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

# Cosine Law

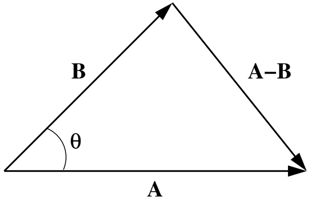



Figure 15: The triangle for the derivation of scalar product.

- The formula for the scalar product $\mathbf{A} \cdot \mathbf{B}$:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$$

# Cosine Law

- Using Cosine Law, we obtain

$$
\begin{aligned}
A^2 + B^2 - 2AB\cos\theta &= |\mathbf{A} - \mathbf{B}|^2 \\
&= (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) \\
&= \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} \\
&= A^2 - 2\mathbf{A} \cdot \mathbf{B} - B^2 \\
\Rightarrow \qquad -2AB\cos\theta &= -2\mathbf{A} \cdot \mathbf{B} \\
AB\cos\theta &= \mathbf{A} \cdot \mathbf{B} \qquad \surd
\end{aligned}
$$

# Projections

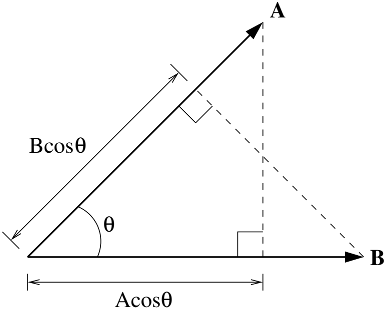



Figure 16: An interpretation of scalar product.

- $A\cos\theta$ is the projection of **B** on **A**; similarly, $B\cos\theta$ is the projection of **B** on **A**.
- The scalar product **A** · **B** is numerically amounts to the length of **B** times the projection of **A** on **B**, and also amounts to the length of **A** times the projection of **B** on **A**.

# Projections

- If $\theta < 90^\circ$, the projections are *positive*; and if $\theta > 90^\circ$, they are *negative*.
- The two vectors are *perpendicular*, i.e., $\theta = 90^\circ$, their scalar products is *zero*.
- If two vectors **A** and **B** *parallel*, i.e., $\theta = 0$, their scalar product is *AB*,
- if they are antiparallel, i.e., $\theta = 180^\circ$, the corresponding scalar product is equal to $-AB$.
- In conclusion, their scalar products vary between $-AB$ and $AB$:

$$-AB \leq \mathbf{A} \cdot \mathbf{B} \leq AB$$

# The Vector Product of Two Vectors

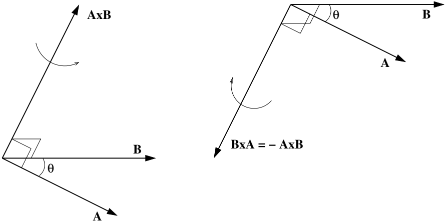



Figure 17: The vector products $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$.

**Condition 1.** The vector product of two vectors **A** and **B**, written $\mathbf{A} \times \mathbf{B}$ results in the unique *vector* whose magnitude is

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

- The other common names for the vector product are the **cross product** (because of the symbol "$\times$" in its notation) and the **outer product**.

# The Vector Product of Two Vectors

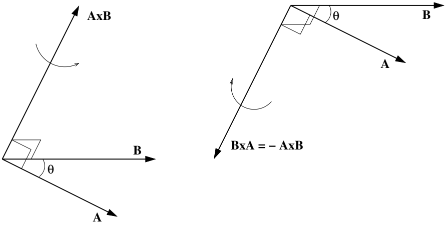


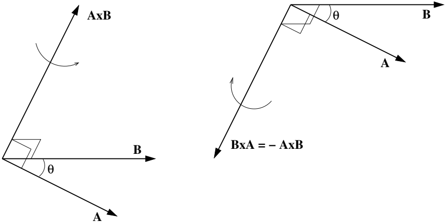

Figure 18: The vector products **A** $\times$ **B** and **B** $\times$ **A**.

**Condition 2.** The vector **A** $\times$ **B** is perpendicular to the plane spanned by **A** and **B**,

$$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$$

where **0** is the *null vector*.

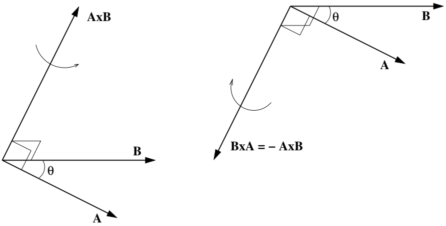
# The Vector Product of Two Vectors




**Condition 3.** The direction of **A** × **B** is determined according to the **right–hand rule**:

- **A**, **B** and **A** × **B** form a **right–handed triad.**

# The Vector Product of Two Vectors



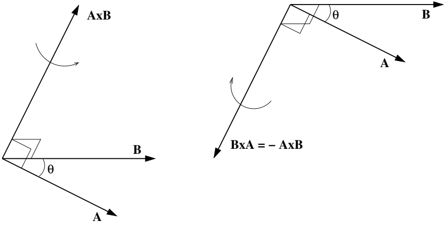

- It directly follows from the equation $|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$ that if two vectors **A** and **B** are parallel, their vector product is the *null vector*:

$$\mathbf{A} \times \mathbf{B} = \mathbf{0} \quad \text{if} \quad \mathbf{A} \parallel \mathbf{B}$$

- The two vectors are perpendicular:

$$|\mathbf{A} \times \mathbf{B}| = AB \quad \text{if} \quad \mathbf{A} \perp \mathbf{B}$$

# Anticommutativity




- The vector product is **anticommutative**:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

- The order of vectors in a vector product is vital.

# Non-associativity

- Consider the **triple cross product** $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
- The vector product is, in general, *not* associative with respect to the vector product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

- However, the vector product *is* associative with respect to the multiplication of a constant $c$:

$$c\,(\mathbf{A} \times \mathbf{B}) = (c\,\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (c\,\mathbf{B})$$

# Distributivity

- The vector product is **distributive**:

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ (\mathbf{A} + \mathbf{B}) \times \mathbf{C} &= \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}\end{aligned}$$

- This can be verified by geometrical consideration and also by some algebraic manipulations.

# Area of Parellelogram

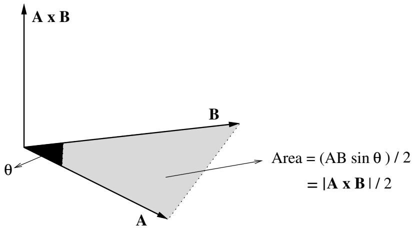



Figure 19: Area of the triangle $= \frac{1}{2}|\mathbf{A} \times \mathbf{B}|$.

- From the definition $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$, the magnitude $|\mathbf{A} \times \mathbf{B}|$ is equal to twice the area of the triangle whose coterminous sides are by the vectors **A** and **B**.
- The magnitude $|\mathbf{A} \times \mathbf{B}|$ is equal to the area of the parallelogram spanned by **A** and **B**.

# The Vector Product of Standard Basis Vectors

- By condition (1), the vector product of any two parallel vectors is zero. As an example, the vector product of a vector with itself is the *null vector*:

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

- First result for the standard basis vectors (SBV)

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

- Another set of identities for SBV follows from condition (3):

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} & \quad \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} & \quad \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} & \quad \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \end{aligned}$$

# Mnemonics for Vector Products of SBV

$$\ldots \, \mathbf{i} \, \mathbf{j} \, \mathbf{k} \, \mathbf{i} \, \mathbf{j} \, \mathbf{k} \, \ldots$$

- The vector product of a vector (say, **i**) with its *right neighbor* (**j**) is the next following vector (**k**) when reading to the right.
- The vector product of a vector (**i**) with its *left neighbor* (**k**) is the next following vector ($-$**j**) when reading to the left.

# Components of $\mathbf{A} \times \mathbf{B}$

- The components of the vector product $\mathbf{A} \times \mathbf{B}$ are deduced by using the mnemonics for SBV:

$$\mathbf{A}\times\mathbf{B} = (A_y B_z - A_z B_y)\,\mathbf{i} + (A_z B_x - A_x B_z)\,\mathbf{j} + (A_x B_y - A_y B_x)\,\mathbf{k}$$

- Since this is an asymmetric equation, it is hard to remember.
- We realize the vector product $\mathbf{A} \times \mathbf{B}$ as the determinant the following $3 \times 3$ square matrix:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \end{aligned}$$

# A mnemonics for $\mathbf{A} \times \mathbf{B}$

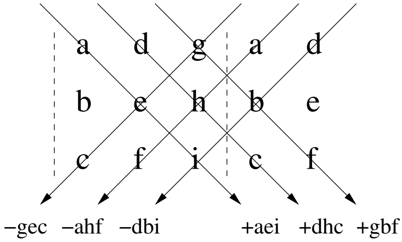



Figure 20: A mnemonics for the evaluation of a $3 \times 3$ square matrix.

- The value of the determinant is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gbf - gec - ahf - dbi$$

# Equations involving vector products

- The following properties of the vector product can now be easily verified using mnemonics for $\mathbf{A} \times \mathbf{B}$:

$$
\begin{aligned}
\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) = 0 \\
\mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} \\
c\,(\mathbf{A} \times \mathbf{B}) &= (c\,\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (c\,\mathbf{B}) \\
\mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\
(\mathbf{A} + \mathbf{B}) \times \mathbf{C} &= \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} \\
\mathbf{A} \times \mathbf{A} &= \mathbf{0}
\end{aligned}
$$